

ACTIONS OF MAXIMAL GROWTH OF HYPERBOLIC GROUPS

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ABSTRACT. We prove that every non-elementary hyperbolic group G acts with maximal growth on some set X such that every orbit of any element $g \in G$ is finite. As a side-product of our approach we prove that for a non-elementary hyperbolic group G and a quasiconvex subgroup of infinite index $\mathcal{H} \leq G$ there exists $g \in G$ such that $\langle \mathcal{H}, g \rangle$ is quasiconvex of infinite index and is isomorphic to $\mathcal{H} * \langle g \rangle$ if and only if $\mathcal{H} \cap E(G) = \{e\}$, where $E(G)$ is the maximal finite normal subgroup of G .

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1. INTRODUCTION

The notion of growth of algebraic structures has been extensively studied. In the case of groups, there are three main classes of growth: polynomial, subexponential, exponential. In [BO] the authors discuss the notion of growth of actions of a group (monoid, ring) on a set (vector space). Let us denote the growth function of a transitive action of a group G generated by S on a set X with respect to some base point $o \in X$ by $g_{o,S}(n) = \#\{o' = og \mid |g| \leq n\}$ (see section 3).

A distinguished class of actions defined and studied in [BO] is that of actions of maximal growth. We observe that in case of a non-amenable group G the growth of action of G on X is *maximal* if there exists $c_1 > 0$ such that

$$c_1 f(n) \leq g_{o,S}(n)$$

for every natural n , where $f(n)$ is the growth of the group G itself (see remark 3.4).

In [BO] the authors construct some examples of actions by the free group of maximal growth and satisfying additional properties, see for example corollary 3.7. The main result of our paper is the following broadening of the aforementioned corollary:

Theorem 1.1. *Let G be a non-elementary hyperbolic group. Then there exist a set X and an action of G on X such that the growth of this action is maximal and each orbit of action by every element $g \in G$ is finite.*

One can observe that the above result follows from theorem 3.8 in this paper.

The main theorem stems from the technical result (theorem) 4.6 which also allows us to generalize and strengthen the result of Arzhantseva [Arzh] conjectured by M. Gromov [Gro].

The following theorem and corollary generalize theorem 1 in [Arzh] by removing the requirement on the hyperbolic group to be torsion-free and formulating necessary and sufficient conditions. We recall the notation $E(g)$ – the maximal elementary subgroup of hyperbolic group G containing g (it exists whenever g is of infinite order, see section 2). Recall also that there exists a unique finite maximal normal subgroup $E(G)$ in every non-elementary hyperbolic group G . We will call $E(G)$ the *finite radical*¹ of G .

Theorem 1.2. *Let G be a non-elementary hyperbolic group and \mathcal{H} be a quasiconvex infinite index subgroup of G .*

¹the term proposed by A. Olshanskiy.

(1) Consider an element x in G of infinite order. There exists a natural number $t > 0$ such that the subgroup $\langle \mathcal{H}, x^t \rangle$ (i.e. generated by \mathcal{H} and x^t) is isomorphic to $\mathcal{H} * \langle x^t \rangle$ if and only if $E(x) \cap \mathcal{H} = \{e\}$.

(2)² An element x , satisfying part (1), exists if and only if $\mathcal{H} \cap E(G) = \{e\}$.

(3) If $\mathcal{H} \cap E(G) = \{e\}$ then for x and t described in part (1) the subgroup $\langle \mathcal{H}, x^t \rangle$ is quasiconvex of infinite index and the intersection $E(G) \cap \langle \mathcal{H}, x^t \rangle$ is trivial.

Part (1) of theorem 1.2 follows also from a more general statement in [M-P](Corollary 1.12) and a particular case when $E(x) = E^+(x)$ appears in theorem 5 [Min]. We also formulate the following (somewhat more general) result concerning arbitrary quasiconvex subgroups of infinite index.

Corollary 1.3. *Let G be a non-elementary hyperbolic group and \mathcal{H} be a quasiconvex subgroup of infinite index in G . Then there exists $g \in G$ of infinite order such that $\langle \mathcal{H} \cdot E(G), g \rangle \cong \mathcal{H} \cdot E(G) *_{E(G)} \langle g, E(G) \rangle$. Moreover $\langle \mathcal{H} \cdot E(G), g \rangle$ is a quasiconvex subgroup of infinite index.*

2. HYPERBOLIC SPACES AND HYPERBOLIC GROUPS

Hyperbolic Spaces. We recall some definitions and properties from the founding article of Gromov [Gro] (see also [Ghys]). Let $(X, | \cdot |)$ be a metric space. We sometimes denote the distance $|x - y|$ between $x, y \in X$ by $d(x, y)$. We assume that X is geodesic, i.e. every two points can be connected by a geodesic line. We refer to a geodesic between some point x, y of X as $[x, y]$. For convenience we denote by $|x|$ the distance $|x - y_0|$ to some fixed point y_0 (usually the identity element of the group).

For a path γ in X we denote the initial (terminal) vertex of γ by γ_- (γ_+), denote by $\|\gamma\|$ the length of path γ and by $|\gamma|$ the distance $|\gamma_+ - \gamma_-|$. Recall that if $0 < \lambda \leq 1$ and $c \geq 0$ then a path γ in X is called (λ, c) -*quasigeodesic* if for every subpath γ_1 of γ the following inequality is satisfied:

$$\|\gamma_1\| \leq \frac{1}{\lambda} |\gamma_1| + c.$$

We call the path γ *geodesic up to c* , if it is $(1, c)$ -quasigeodesic.

Define a scalar (Gromov) product of x, y with respect to z by formula

$$(x, y)_z = \frac{1}{2}(|x - z| + |y - z| - |x - y|).$$

An (equivalent) implicit definition of the Gromov product illustrates its geometric significance:

$$\begin{aligned} (1) \quad & (x, y)_z + (x, z)_y = |z - y|; \\ & (x, y)_z + (y, z)_x = |z - x|; \\ & (y, z)_x + (z, x)_y = |x - y|. \end{aligned}$$

A space X is called δ -hyperbolic if there exists a non-negative integer δ such that the following inequality holds:

$$(H1) \quad \forall x, y, z, t \in X, \quad (x, y)_z \geq \min\{(x, t)_z, (y, t)_z\} - \delta.$$

The condition (H1) implies (and in fact is equivalent up to constant) the following:

(H2) *For every triple of points x, y, z in X every geodesic $[x, y]$ is within the (closed) 4δ -neighborhood of the union $[x, z] \cup [y, z]$.*

²When preparing this paper for publication the author learned that a version of this statement have been presented by F. Dudkin and K. Sviridov on a Group Theory seminar in IM SORAN (Novosibirsk) in November, 2011.

(H3) For every four points x, y, z, t in X we have $|x - y| + |z - t| \leq \max\{|x - z| + |y - t|, |x - t| + |y - z|\} + 2\delta$.

Hyperbolic Groups. Let G be a finitely presented group with presentation $gp(S|\mathcal{D})$. We assume that no generator in S is equal to e in G . We consider G as a metric space with respect to the distance function $|g - h| = |g^{-1}h|$ for every g and h . We denote by $|g|$ the length of a minimal (geodesic) word with respect to the generators S equal to g . The notation (g, h) is the Gromov product $(g, h)_e$ with respect to the identity vertex e .

We denote the (right) Cayley graph of the group by $Cay(G)$. The graph $Cay(G)$ has a set of vertices G , and a pair of vertices g_1, g_2 is connected by an edge of length 1 labeled by s if and only if $g_1^{-1}g_2 = s$ in G for some $s \in S^{\pm 1}$. It is clear that $Cay(G)$ may be considered as a geodesic space: one identifies every edge of $Cay(G)$ with interval $[0, 1]$ and chooses the maximal metric d which agrees with metric on every edge. Define a label function on paths in $Cay(G)$. From now on, by a path in $Cay(G)$ we mean a path $p = p_1 \dots p_n$, where p_i is an edge in $Cay(G)$ between some group elements g_i, g_{i+1} for every $1 \leq i \leq n$. A label $lab(p)$ function is defined on any path p by equality $lab(p) = lab(p_1) \dots lab(p_n)$, i.e. $lab(p)$ is a word in alphabet $S^{\pm 1}$.

Hence a unique word $lab(p)$ is assigned to a path p in $Cay(G)$. On the other hand for every word w in alphabet $S^{\pm 1}$ there exists a unique path p in $Cay(G)$ starting from the identity vertex with label w . Hence there is a one-to-one correspondence between paths with initial vertex e (the identity vertex in G) and words in alphabet $S^{\pm 1}$, so we will not distinguish between a word in the alphabet $S^{\pm 1}$ and its image in $Cay(G)$, i.e. a path starting from the identity vertex. Thus, when considering some words X, Y, Z in the alphabet $S^{\pm 1}$, we can talk about the path $\gamma = XYZ$ in the Cayley graph of G originating in the identity vertex e . To distinguish a path Y with initial vertex e from the subpath of γ with label Y we denote the latter as ${}_\gamma Y$. We will talk about values $|X|, \|X\|$ for a word X in alphabet $S^{\pm 1}$ meaning these values on the corresponding paths in $Cay(G)$. Given elements x_1, \dots, x_k in G we may write $lab(p) = x_1^{t_1} \dots x_k^{t_k}$ for some path p in $Cay(G)$, $t_i \in \mathbb{Z}$ if for some geodesic words X_1, \dots, X_k representing elements x_1, \dots, x_k we have $lab(p) = X_1^{t_1} \dots X_k^{t_k}$.

For a point x in a metric space X and $r \geq 0$ we denote by $B_r(x)$ a metric ball of radius r around x . For a set $D \subset X$ we denote by $B_r(D)$ a (closed) r -neighborhood of D in X (i.e. $B_r(D) = \cup_{x \in D} B_r(x)$). We denote the ball $B_R(e)$ in the Cayley graph $Cay(G)$ by B_R . Given a set $D \in Cay(G)$ we denote by $\# \{D\}$ a number of vertices in D .

A group G is called δ -hyperbolic for some $\delta \geq 0$, if its Cayley graph is δ -hyperbolic. It is well known that hyperbolicity does not depend on choice of a finite presentation of the group G (while δ does depend on presentation).

Lemma 2.1. ([Gro], [Ghys] p. 87) *There exists a constant $H = H(\delta, \lambda, c)$ such that for any (λ, c) -quasigeodesic path p in a δ -hyperbolic space and any geodesic path q with conditions $q_- = p_-$ and $q_+ = p_+$, the paths p and q are within (closed) H -neighborhoods of each other.*

We recall that a (sub)group is called elementary if it contains a cyclic group of finite index. For any element $g \in G$ of infinite order in a hyperbolic group, there exists a unique maximal elementary subgroup $E(g)$ containing g (see [Gro], [Olsh93] lemma 1.16). It is well known that for a hyperbolic group G

$$E(g) = \{x \in G | \exists n \neq 0 \text{ such that } xg^n x^{-1} = g^{\pm n} \text{ in } G\},$$

and if a is an element in $E(g)$ of infinite order then $E(g) = E(a)$. We recall also that if G is a non-elementary hyperbolic then the subgroup $E(G) = \cap_g \{E(g) | g \in G, \text{ order of } g \text{ is infinite}\}$ is a unique maximal finite normal subgroup ([Olsh93], prop.1). As agreed in the introduction, we will call $E(G)$ the *finite radical* of a non-elementary group G .

Definition 2.2. A subset A is called K -quasiconvex in the metric space X if for any pair of points $a, b \in A$ every geodesic connecting a and b (in X) is within (closed) K -neighborhood of A . A subgroup \mathcal{H} of a hyperbolic group G is K -quasiconvex if it forms a K -quasiconvex subset in the graph $\text{Cay}(G)$.

It is said that \mathcal{H} is quasiconvex if it is K -quasiconvex for some $K \geq 0$. Note also that the left multiplication $g \rightarrow ag$ induces an isometry of G and hence, for a K -quasiconvex subgroup \mathcal{H} , the right coset $a\mathcal{H}$ is K -quasiconvex for any a in G .

Lemma 2.3. ([GMRS], lemma 1.2) Let H be a K -quasiconvex subgroup of a δ -hyperbolic group G . If the shortest representative of a double coset HgH has length greater than $2K + 2\delta$, then the intersection $H \cap g^{-1}Hg$ consists of elements shorter than $2K + 8\delta + 2$ and, hence, is finite.

Proposition 2.4. ([Arzh], prop.1) Let G be a word hyperbolic group and H a quasiconvex subgroup of G of infinite index. Then the number of double cosets of G modulo H is infinite.

We quote the following:

Theorem 2.5. ([Mack], theorem 6.4) Let G be a hyperbolic group and H be a quasiconvex subgroup of infinite index in G . Then there exist $C > 0$ and a set-theoretic section $s : G/H \rightarrow G$ such that:

- (1) the section s maps each coset gH to an element $g' \in gH$ with $|g'|$ minimal among all representatives in gH ;
- (2) the group G is within C -neighborhood of $s(G/H)$.

The following lemma summarize some properties of elementary subgroups of hyperbolic groups ([Gro]; [Ghys] p.150, p.154; [CDP] Pr. 4.2, Ch.10; [Olsh93] lemma 2.2).

Lemma 2.6. Let G be a hyperbolic group.

- (i) For any word W of infinite order in the hyperbolic group G there exist constants $0 < \lambda \leq 1$ and $c \geq 0$ such that any path with label W^m in $\text{Cay}(G)$ is (λ, c) -quasigeodesic for any m .
- (ii) Let E be an infinite elementary subgroup in G . Then there exists a constant $K = K(E) \geq 0$ such that the subgroup E is K -quasiconvex.
- (iii) If W is a geodesic word and p is a path with label W^n then there exists K (independent of n) such that the path p and the geodesic $[p_-, p_+]$ are within K -neighborhoods of each other.
- (vi) Let g, h be elements of infinite order such that $E(g) \neq E(h)$. Then the Gromov products $(g^m, h^n), (g^u, g^v), (h^u, h^v)$ are bounded by some constant C depending on g, h only provided $uv < 0$.

Following [Olsh93], we call a pair of elements x, y of infinite order in G non-commensurable if x^k is not conjugate to y^s for any non-zero integers k, s .

Lemma 2.7. ([Olsh93], lemmas 3.4, 3.8) There exist infinitely many pairwise non-commensurable elements g_1, g_2, \dots in a non-elementary hyperbolic group G such that $E(g_i) = \langle g_i \rangle \times E(G)$ for every i .

Let W be a word, and let us fix some factorization $W \equiv W_1^{i_1} W_2^{i_2} \dots W_k^{i_k}$ for some words W_1, \dots, W_k . Consider a path q with label W in $\text{Cay}(G)$.

Consider all vertices o_i which are the terminal vertices of initial subpaths p_i of q such that $\text{lab}(p_i) = W_1^{i_1} \dots W_{m-1}^{i_{m-1}} W_m^s$, where $m \leq k$ and $s = 0, \dots, i_m$. Following [Olsh93], we call vertices $\{o_i\}$ phase vertices of q relative to factorization $W_1^{i_1} W_2^{i_2} \dots W_k^{i_k}$ of the $\text{lab}(q)$. We enumerate distinct phase vertices along the path q starting from $o_0 = q_-$; the total number of such vertices is $(|i_1| + \dots + |i_k| + 1)$.

Assume we have a pair of paths q, \bar{q} in $\text{Cay}(G)$ with phase vertices o_i and \bar{o}_j where $i = 1, \dots, l$, $j = 1, \dots, m$ for some positive integers l, m . We call a shortest path between a phase vertex o_i and some phase vertex \bar{o}_j of \bar{q} a *phase path* with initial vertex o_i . We may also talk about phase vertices of subpaths p of q meaning these vertices o_i which belong to p .

Definition 2.8. [Olsh93] *Let the words W_1, \dots, W_l represent some elements of infinite order in G . Fix some $A \geq 0$ and an integer m to define a set $S_m = S(W_1, \dots, W_l, A, m)$ of words*

$$W = X_0 W_1^{m_1} X_1 W_2^{m_2} \dots W_l^{m_l} X_l \text{ where } |m_2|, \dots, |m_{l-1}| \geq m,$$

such that $\|X_i\| \leq A$ for $i = 0, \dots, l$ and $X_i^{-1} W_i X_i \notin E(W_{i+1})$ in G for $i = 1, \dots, l-1$. If $l = 1$ we assume that $|m_1| \geq m$.

Lemma 2.9. ([Olsh93], lemma 2.4) *There exist $\lambda > 0$, $c \geq 0$ and $m > 0$ (depending on K, W_1, W_2, \dots, W_l) such that any word $W \in S_m$ is (λ, c) -quasigeodesic. If $W_i \equiv W_j$ for all i, j then the constant λ does not depend on A, l .*

Consider a closed path $p_1 q_1 p_2 q_2$ in $\text{Cay}(G)$. Let $q_1 = x_1 t_1 x_2 t_2 \dots x_l t_l$ where $\text{lab}(x_i) = X_i$ and $\text{lab}(t_i) = W_i^{m_i}$ for some $W = X_0 W_1^{m_1} X_1 W_2^{m_2} \dots W_l^{m_l} X_l \in S_m$. Similarly, we let $q_2^{-1} = \bar{x}_1 \bar{t}_1 \dots \bar{x}_l \bar{t}_l$ where $\text{lab}(\bar{x}_i) = \bar{X}_i$ and $\text{lab}(\bar{t}_i) = \bar{W}_i^{\bar{m}_i}$ for some $\bar{W} = \bar{X}_0 \bar{W}_1^{\bar{m}_1} \bar{X}_1 \bar{W}_2^{\bar{m}_2} \dots \bar{W}_l^{\bar{m}_l} \bar{X}_l \in \bar{S}_m$. Define phase vertices o_i and \bar{o}_j on q_1 and q_2 relative to factorizations $X_0 W_1^{m_1} X_1 W_2^{m_2} \dots W_l^{m_l} X_l$ and $\bar{X}_0 \bar{W}_1^{\bar{m}_1} \bar{X}_1 \bar{W}_2^{\bar{m}_2} \dots \bar{W}_l^{\bar{m}_l} \bar{X}_l$. As in [Olsh93], We say that paths t_i and \bar{t}_j are compatible if there exists a phase path v_i with $\text{lab}(v_i) = V_i$ between a phase vertex of t_i and \bar{t}_j such that there exist natural numbers a, b satisfying $(V_i \bar{W}_j V_i^{-1})^a = W_i^b$.

Lemma 2.10. ([Olsh93], lemma 2.5) *Provided the conditions for q_1 and q_2 hold, and $|p_1|, |p_2| < C$ for some C , there exists an integer m and an integer k , where $|k| \leq 1$ such that t_i and \bar{t}_{i+k} are compatible for any $i = 2, \dots, l-1$ provided that $|m_2|, \dots, |m_{l-1}|, |\bar{m}_2|, \dots, |\bar{m}_{l-1}| \geq m$ and for $i = 1$ (resp. $i = l$) if $|m_1| \geq m$, (resp. $|m_l| \geq m$). Moreover t_i is not compatible with \bar{t}_j if $j \neq i + k$.*

3. ACTIONS OF MAXIMAL GROWTH

Let G be a group generated by a finite set S and suppose that G acts on a set X from the right:

$$xe = x, \quad (xg_1)g_2 = x(g_1g_2) \text{ in } G \text{ for all } x \in X; g_1, g_2 \in G.$$

We assume that the action is transitive (i.e. $X = oG$, where o is some element from X). Consider the set $B_n(o)$ of elements $og \in X$ such that $g \in G$ and $|g| \leq n$. Then the growth function of the right action of G on X is $f_{o,S}(n) = \# \{B_n(o)\}$. Let $o' = og_0 \in G$ and denote $|g_0|$ by C . It is clear that $B_n(o') \subset B_{n+C}(o)$ and hence

$$f_{o,S}(n+C) \geq f_{o',S}(n).$$

Consider a set \mathcal{F} of functions from \mathbb{N}_0 to \mathbb{N}_0 . A pair $f, g \in \mathcal{F}$ is said (see [BO], §1.4) to satisfy the relation $f \prec g$ if there exist a non-negative integer C such that $f(n) \leq g(n+C)$ for every $n \in \mathbb{N}_0$. Clearly the relation \prec is transitive and reflexive. Functions $f, g \in \mathcal{F}$ are said to be *equivalent* ([BO], §1.4) if $f \prec g$ and $g \prec f$. According to the discussion above, growth functions of transitive action of G on X with respect to different base points o, o' are equivalent.

If the group G acts from the right on $X = G$, we get the usual growth function and denote it by $f(n)$; clearly the growth of any action of G is bounded by the usual growth function of G : $f_{o,S}(n) \leq f(n)$ for any $o \in X$.

If H is a stabilizer of o , then every element $x \in X$ is in one-to-one correspondence with a coset Hg in G such that $x = og$ and the right actions of G on X and on $H \backslash G$ are isomorphic.

Definition 3.1. ([BO], §2) *Let $f(n)$ be a growth function of G relative to a finite generating set S and consider a transitive action of G on a set X . Then the growth of the action is called maximal if the function $f_{o,S}(n)$ is equivalent to $f(n)$.*

In this paper we discuss the growth of actions of hyperbolic groups which are known to be non-amenable (see remark 3.2). We recall that a group G is called amenable if there exists a finitely additive left invariant probability measure on G (see [Gre]).

Remark 3.2. *Every non-elementary hyperbolic group is non-amenable.*

Proof If G is a non-elementary hyperbolic group, then it contains a free non-cyclic subgroup F_2 ([Gro], [Ghys], p. 157). But a free group of rank greater than one is non-amenable (see [Gre] 1.2.8). On the other hand, a subgroup of an amenable group is amenable ([Gre], theorem 1.2.5). Hence G cannot be amenable. \square

The famous Følner amenability criterion ([Gre]) yields the following:

Corollary 3.3. *For every non-elementary hyperbolic group G there exists $\epsilon > 0$ (depending on G only) such that $\#\{B_{R+1}\} \geq (1 + \epsilon)\#\{B_R\}$ for any R .*

Remark 3.4. *Let G be a non-amenable group with growth function $f(x)$ relative to a finite generating set S . Assume G acts on X with respect to some base point $o \in X$; denote the growth function of this action by $g_{o,S}(x)$. Then there exists $c_1 > 0$ such that the inequality $g_{o,S}(n) \geq c_1 f(n)$ holds for all natural n if and only if the action has maximal growth.*

Proof We first show the "only if" part. By corollary 3.3 there exists $\epsilon > 0$ such that the recursive formula $f(n+1) \geq (1 + \epsilon)f(n)$ holds for every n . We choose a natural C satisfying $c_1(1 + \epsilon)^C \geq 1$. Then, applying the recursive formula C times we get:

$$g_{o,S}(n + C) \geq c_1 f(n + C) \geq c_1(1 + \epsilon)^C f(n) \geq f(n).$$

Now assume that the action has maximal growth, i.e. $g_{o,S}(n + C) \geq f(n)$ for some natural number $C \geq 0$ and every natural n . It is clear from definition of $g_{o,S}$ that $g_{o,S}(n + C) \leq (2\#\{S\})^C \times g_{o,S}(n)$ and hence for $c_1 = (2\#\{S\})^{-C}$ the inequality $g_{o,S}(n) \geq c_1 f(n)$ holds. \square

We recall the notion of exponential growth rate of a group G with respect to the set of generators S : $\lambda(G, S) = \lim_{n \rightarrow \infty} \sqrt[n]{f(n)}$, where $f(n)$ is a growth function of G .

Let S be a finite generating set in G and let N be an infinite normal subgroup of G . We denote the image of S under the canonical homomorphism $G \rightarrow G/N$ by \bar{S} . The following theorem is often summarized by saying that the hyperbolic groups are "growth tight":

Theorem 3.5. [AL] *Let G be a non-elementary hyperbolic group and S any finite set of generators for G . Then for any infinite normal subgroup N of G we have $\lambda(G, S) > \lambda(G/N, \bar{S})$.*

The next corollary restates the above theorem in terms of maximal growth.

Corollary 3.6. *Assume G is a hyperbolic group acting on some set X from the right with maximal growth. Then the kernel of this action is a finite normal subgroup.*

Proof Let N be the kernel of the action on X . For any point $o \in X$ we have that $oNg = og$ for all $g \in G$ and hence the growth function of the action $g_{o,S}$ satisfies:

$$(2) \quad g_{o,S}(n) \leq f_{G/N}(n) \text{ for every } n \in \mathbb{N},$$

where $\bar{f}(n)$ is the growth function of G/N with respect to images \bar{S} of generators S of G . If $f(n)$ is the growth function of G with respect to S and the growth of the action is maximal,

then there exists $c_1 > 0$ such that $g_{o,S}(n) \geq c_1 f(n)$ for every $n \in \mathbb{N}$. Hence, using (2) and the last inequality, we have:

$$\lambda(G/N, \bar{S}) = \lim_{n \rightarrow \infty} \sqrt[n]{f(n)} \geq \lim_{n \rightarrow \infty} \sqrt[n]{g_{o,S}(n)} \geq \lim_{n \rightarrow \infty} \sqrt[n]{c_1 f(n)} = \lambda(G, S),$$

which by Theorem 3.5 can only hold when N is finite. \square

In [BO] the authors provide examples of maximal growth actions of free groups satisfying some additional properties.

Recall that in [Sta] a subgroup H of a group G is said to satisfy the Burnside condition if for any $a \in G$ there exists a natural number $n \neq 0$ such that a^n is in H . One of the main results of the aforementioned paper is the following:

Corollary 3.7. ([BO], corollary 6) *Any finitely generated subgroup H of infinite index in the free group F of rank greater than 1 is contained as a free factor in a free subgroup K satisfying the Burnside condition. One can choose K with maximal growth of action of F on $K \backslash F$. It follows that there exists a transitive action of F , with maximal growth and with finite orbits for each element $g \in F$, which factors through the action of F on $H \backslash F$.*

The following theorem generalizes the corollary 6 of [BO] from free groups to non-elementary hyperbolic ones:

Theorem 3.8. *Let G be a non-elementary hyperbolic group with growth function $f(n)$. Then for any $0 < q < 1$ there exists a free subgroup H in G satisfying the Burnside condition and such that the growth $f_{H \backslash G}(n)$ of right action of G on $H \backslash G$ satisfies $f_{H \backslash G}(n) \geq q f(n)$. In particular, the growth of such action is maximal.*

Consequently, for every non-elementary hyperbolic group G there exists a transitive action of G with maximal growth such that the orbit of action of any element $g \in G$ is finite.

Throughout this paper we will mainly discuss properties of left cosets. The connection between the right and left cosets is established by the following observation:

Remark 3.9. *The right coset Hg intersects the ball B_R in G if and only if the left coset $g^{-1}H$ intersects B_R . \square*

The abundance of actions of maximal growth is evident from the following:

Corollary 3.10. *Let G be a hyperbolic group and H be a quasiconvex subgroup of infinite index in G . Then the natural right action of G on $H \backslash G$ has maximal growth.*

Proof We first consider left cosets G/H . By theorem 2.5 there exists $C > 0$ and the section s such that the group G is in $B_C(s(G/H))$. Hence for every $g \in B_R$ there exists $\bar{g} \in s(G/H)$ such that $|g - \bar{g}| \leq C$. By definition of s , $|\bar{g}| \leq |g|$ and thus $\bar{g} \in B_R$. We get that $B_R \subset \cup_{g \in B_R \cap s(G/H)} B_C(g)$, which implies:

$$(3) \quad f(R) = \#\{B_R\} \leq \#\{B_C\} \times \#\{s(G/H) \cap B_R\}.$$

If $g_1, g_2 \in s(G/H) \cap B_R$ then (because the map s is a section) $g_1 H \neq g_2 H$. We get that $\#\{s(G/H) \cap B_R\} \leq \#\{gH | gH \cap B_R \neq \emptyset\}$ and the remark 3.9 provides $\#\{gH | gH \cap B_R \neq \emptyset\} = \#\{Hg^{-1} | Hg^{-1} \cap B_R \neq \emptyset\}$, thus

$$(4) \quad \#\{s(G/H) \cap B_R\} \leq \#\{Hg^{-1} | Hg^{-1} \cap B_R \neq \emptyset\}.$$

Evidently the sets $\{Hg^{-1} | Hg^{-1} \cap B_R \neq \emptyset\}$ and $\{Hg | \exists g_1 : Hg_1 = Hg \text{ \& } |g_1| \leq R\}$ contain the same cosets, and by definition of the growth function $f_{H, G/H}(R)$ of natural right action of G on G/H : $\#\{Hg | \exists g_1 : Hg_1 = Hg \text{ \& } |g_1| \leq R\} = f_{H, G/H}(R)$. Using inequalities (3), (4) and the last equality we get:

$$f(R) \leq \#\{B_C\} \times \#\{s(G/H) \cap B_R\} \leq \#\{B_C\} \times \#\{Hg | \exists g_1 : Hg_1 = Hg \text{ \& } |g_1| \leq R\}$$

$$\leq \#\{B_C\} \times f_{H,G/H}(R).$$

By remark 3.4 the action has maximal growth. \square

4. PROOF OF THEOREM 3.8

Throughout this paragraph we assume that the group G is non-elementary hyperbolic. The following lemma summarizes some geometric properties that we will need later.

Lemma 4.1. *Let a, b, c, d be points in a δ -hyperbolic space X .*

(i) *Assume that $(a, c)_b, (b, d)_c \leq M$. If we take Q such that $(a, d)_b \leq Q$ then $|a - d| \geq |a - b| + |b - c| + |c - d| - 2M - 2Q$. Moreover, if $|b - c| > 2M + \delta$, then we can choose $Q = M + \delta$.*

Assume that the point d is on the segment $[a, b]$ and

(ii) *$d \in B_{M_1}([a, c])$ for some $M_1 \geq 0$. Then*

$$(5) \quad |d - b| \geq (a, c)_b - \delta - M_1 \text{ and}$$

$$(6) \quad |a - c| \geq |a - b| + |b - c| - 2|d - b| - 2\delta - 2M_1.$$

(iii) *that $(b, c)_a - 5\delta > |a - d|$. Then $d \in B_{4\delta}([a, c])$.*

(iv) *the vertex d is at least $D > 0$ away from each point a, b . Then*

$$|d - c| \leq \max\{|a - c|, |b - c|\} + 2\delta - D.$$

Proof (i) By definition of Gromov product and conditions of part (i) we get that

$$\begin{aligned} |a - d| &= |a - b| + |b - d| - 2(a, d)_b = |a - b| + (|b - c| + |c - d| - 2(b, d)_c - 2(a, d)_b) \geq \\ &\geq |a - b| + |b - c| + |c - d| - 2M - 2Q. \end{aligned}$$

It remains to show the second claim in part (i). If $|b - c| > 2M + \delta$, then by (1):

$$(7) \quad (c, d)_b = |b - c| - (b, d)_c > 2M + \delta - M.$$

By inequality (7) and definition (H1) of δ -hyperbolic space we have

$$(c, d)_b > M + \delta \geq (a, c)_b + \delta \geq \min\{(a, d)_b, (c, d)_b\},$$

which implies that $(a, d)_b \leq M + \delta$.

(ii) Let d' be a point on $[a, c]$ at distance at most M_1 from d . Then by (H1) and definitions of d, d' :

$$\begin{aligned} (d, c)_b &= \frac{1}{2}(|b - d| + |b - c| - |c - d|) = \frac{1}{2}(|a - b| - |a - d| + |b - c| - |c - d|) = \\ &= \frac{1}{2}(|a - b| + |b - c| - |a - c|) + \frac{1}{2}(|a - c| - |a - d| - |c - d|) \geq \\ &\geq (a, c)_b + \frac{1}{2}(|a - c| - [|a - d'| + |d - d'|] - [|d - d'| + |c - d'|]) = (a, c)_b - |d - d'|. \end{aligned}$$

We obtain the first claim of part (ii) by using definition (H1) and the expression for $(d, c)_b$ above :

$$|d - b| = (d, a)_b \geq \min\{(d, c)_b, (a, c)_b\} - \delta \geq (a, c)_b - \delta - |d - d'|,$$

and apply it to obtain the second claim:

$$|a - c| = |a - b| + |b - c| - 2(a, c)_b \geq |a - b| + |b - c| - 2(|d - b| + |d - d'| + \delta).$$

(iii) Assume $d \notin B_{4\delta}([a, c])$, then $d \in B_{4\delta}([b, c])$ by (H2). We apply part (ii) to the points b, a, c, d with $M_1 = 4\delta$ and obtain that $|a - d| \geq (b, c)_a - \delta - 4\delta$. Contradiction.

(iv) By definition (H3) we have that

$$|d - c| + |b - a| \leq \max\{|a - c| + |b - d|, |b - c| + |a - d|\} + 2\delta, \text{ hence}$$

$$\begin{aligned}
|d - c| &\leq \max\{|a - c| + (|b - d| - |b - a|), |b - c| + (|a - d| - |b - a|)\} + 2\delta \leq \\
&\leq \max\{|a - c|, |b - c|\} + 2\delta + \max\{|b - d| - |b - a|, |a - d| - |b - a|\} \leq \\
&\leq \max\{|a - c|, |b - c|\} + 2\delta - D. \square
\end{aligned}$$

Lemma 4.2. *Consider subgroups \mathcal{H}_1 and \mathcal{H}_2 in a finitely generated group G . If there exists $M \geq 0$ such that $\# \{B_M(\mathcal{H}_1) \cap B_M(\mathcal{H}_2)\} = \infty$ then $\#\{\mathcal{H}_1 \cap \mathcal{H}_2\} = \infty$.*

The lemma is equivalent to the statement: if $\#\{\mathcal{H}_1 \cap \mathcal{H}_2\} < \infty$ then the set $B_M(\mathcal{H}_1) \cap B_M(\mathcal{H}_2)$ is finite for every non-negative M .

Proof Assume that $\# \{B_M(\mathcal{H}_1) \cap B_M(\mathcal{H}_2)\} = \infty$ for some $M \geq 0$, then there exist infinite sequences of elements $\{h_{1i}\} \subset \mathcal{H}_1$ and $\{h_{2i}\} \subset \mathcal{H}_2$ such that $|h_{1i}^{-1}h_{2i}| = |h_{1i} - h_{2i}| \leq 2M$ for every $i \in \mathbb{N}$. We denote the element $h_{1i}^{-1}h_{2i}$ by l_i . Since $|l_i| \leq 2M$ and the geometry of $\text{Cay}(G)$ is proper, there exists an element $l \in G$ and a subsequence $\{i_j\}$, $j \in \mathbb{N}$ such that $l_{i_{j_1}} = l_{i_{j_2}} = l$ in G for any $j_1, j_2 \in \mathbb{N}$ and thus $h_{1s}^{-1}h_{2s} = h_{1k}^{-1}h_{2k}$ in G for every $s, k \in \{i_j\}$. We obtained that $h_{1k}h_{1s}^{-1} = h_{2k}h_{2s}^{-1}$ belongs to $\mathcal{H}_1 \cap \mathcal{H}_2$ for every $s, k \in \{i_j\}$. For every fixed k , $\lim_{s \rightarrow \infty} |h_{1k}h_{1s}^{-1}| \geq (\lim_{s \rightarrow \infty} |h_{1s}^{-1}|) - |h_{1k}| = \infty$ which implies that the intersection $\mathcal{H}_1 \cap \mathcal{H}_2$ does not belong to a ball B_R for any $R \geq 0$ and thus is infinite. \square

Lemma 4.3. *Let \mathcal{H} be a K -quasiconvex subgroup in a δ -hyperbolic group G and let E be an infinite elementary subgroup in G . Then the following assertions are equivalent:*

- (i) *for any number $M > 0$ we have $E \not\subset B_M(\mathcal{H})$;*
- (ii) *$\# \{E \cap \mathcal{H}\} < \infty$;*
- (iii) *There exists $M > 0$ (depending on E and \mathcal{H} only) such that $(x, h) < M$ for any $x \in E$, $h \in \mathcal{H}$.*

Proof We first show that (ii) implies (i). If the intersection $E \cap \mathcal{H}$ is finite then by lemma 4.2 we have $\# \{B_M(E) \cap B_M(\mathcal{H})\} < \infty$ for any $M \geq 0$. In particular, $\# \{E \cap B_M(\mathcal{H})\} < \infty$ and hence $E \not\subset B_M(\mathcal{H})$ for any $M \geq 0$.

Now we show that (iii) implies (ii). Let x be an element of E and $|x| > 2M$. We have

$$|x - h| = |x| + |h| - 2(x, h) \geq |x| - 2M > 0,$$

hence $x \notin \mathcal{H}$ and so the intersection $E \cap \mathcal{H}$ belongs to the ball $B_{2M}(e)$ which is a finite set.

It remains to show that (i) implies (iii). Since E is infinite virtually cyclic we can choose an element x in E of infinite order and thus E is of finite index in $\langle x \rangle$. Hence

$$(8) \quad \text{there exists a constant } M_0 \text{ such that } E(x) \subset B_{M_0}(\langle x \rangle).$$

Let $K_x = K(\langle x \rangle)$ be a constant provided by lemma 2.6(iii).

Assume (iii) does not hold, i.e. for every $M \geq 0$ there exist $y \in E$, $h \in \mathcal{H}$, satisfying $(y, h) > M + M_0$. Then by (8), $y = x^t a$ for some $a \in E$, $|a| \leq M_0$, $t \neq 0$ and

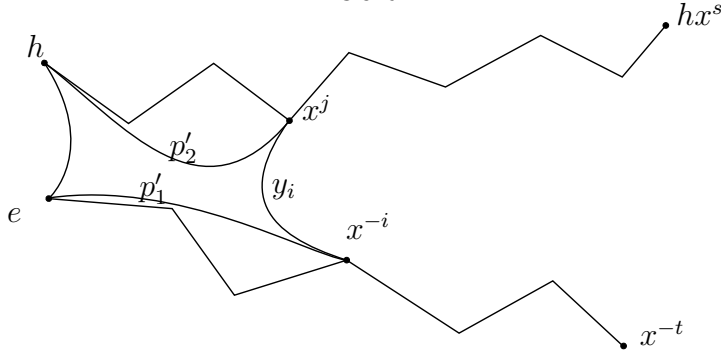
$$(9) \quad (x^t, h) \geq (x^t a, h) - M_0 > M + M_0 - M_0 = M.$$

Hence for every $M > 0$ there exists an integer t and an element $h \in \mathcal{H}$ such that $(x^t, h) > M$.

Now we fix an arbitrary t and choose M so that $|x^t| < M - K_x - 5\delta$. We may assume without loss of generality that $t \geq 0$. Then by (9), there exist $t' \geq t$ and $h \in \mathcal{H}$ such that $(x^{t'}, h) > M$. By lemma 2.6(ii) vertices x^m are within K_x -neighborhood of $[e, x^{t'}]$ for any $0 \leq m \leq t'$. In particular, there exists a vertex $b \in [e, x^{t'}]$ such that $|x^t - b| \leq K_x$ and thus $|b| \leq M - 5\delta < (x^{t'}, h) - 5\delta$. By lemma 4.1(iii), we have that $b \in B_{4\delta}([e, h])$ and, because \mathcal{H} is K -quasiconvex, $b \in B_{4\delta+K}(\mathcal{H})$. Finally, we get that x^t belongs to $B_{4\delta+K+K_x}(\mathcal{H})$ for every t contrary to (i). \square

In lemma 4.4 and theorem 4.6 we follow in part the line of argument from [Arzh] (in particular we apply lemma 13[Arzh]).

FIGURE 1.



Lemma 4.4. *Let x be an element of infinite order in G and choose a constant $M_1 \geq 0$. Then there exist a natural number m and a number $M_2 \geq 0$ such that for any element h in G satisfying conditions $|h| < 2M_1$ and $h \notin E(x)$ and any $|t|, |s| \geq m$ the following inequality holds:*

$$|x^t h x^s| \geq |x^t| + |h| + |x^s| - M_2.$$

Proof For a pair of integers s, t we consider a closed path $p_1 q_1 p_2 q_2$ in $\text{Cay}(G)$, where the path p_1 starts from e and $\text{lab}(p_1) = x^{-t}$, the path q_1 is geodesic and ends at vertex $h x^s$, the path p_2 satisfies $\text{lab}(p_2) = x^{-s}$ and q_2 is geodesic with $\text{lab}(q_2) = h^{-1}$. We define phase vertices a_i on p_1 and phase vertices b_j on p_2 relative to the natural factorizations x^{-t} and x^s respectively ($i = 0, \dots, -t$ and $j = 0, \dots, s$).

Step 1. We take constants $\lambda, c, K_x = K(\langle x \rangle)$ provided by lemma 2.6 for the cyclic group $\langle x \rangle$ and define

$$C = \max\{2K_x + \frac{1}{2}|x|, K_x + 2M_1\} + 8\delta.$$

Let us denote by y_i a phase path connecting vertex a_i with some phase vertex b_j of p_2 . Assume that $|y_i| \leq C$ for some i . We define subpaths p'_1, p'_2 of paths p_1, p_2 , where the path p'_1 connects a_0 to a_i and p'_2 connects b_j to b_0 . Considering the closed path $p'_1 y_i p'_2 q_2$, we have

$$|j||x| \geq |h - h x^j| \geq |x^j| - |h| - |y_i| \geq \lambda|x||i| - c - 2M_1 - C,$$

which implies that $|j| \geq \frac{\lambda|i||x| - c - 2M_1}{|x|} \geq \lambda|i| - c_1$, where $c_1 = \frac{c + C + 2M_1}{|x|}$.

Since we have fixed the constant C , we may apply lemma 2.10 to the closed path $p'_1 y_i p'_2 q_2$ to obtain an integer m_0 such that if we choose a number i_0 satisfying $\lambda|i_0| - c_1 \geq m_0$ and hence $|i_0| \geq m_0$ then there exists a phase path $y_{i'}$, $|i'| \leq i_0$ such that $\text{lab}(y_{i'}) \in E(x)$. If the vertex $b_{j'}$ is the end vertex of $y_{i'}$, we get that $x^{-i'} \text{lab}(y_{i'}) x^{-j'} h = e$ in G and hence $h \in E(x)$, contradiction. We obtained that there exist i_0 depending on x and C , such that

$$(10) \quad |y_{i_0}| > C.$$

Step 2. We show now that $a_{i_0} \in B_{8\delta + K_x}(q_1)$. By lemma 2.6 $a_i \in B_{K_x}([e, x^{-t}])$ and using twice the condition (H2), we get that a_{i_0} belongs to $B_{8\delta + K_x}(q_2 \cup [h, h x^s] \cup q_1)$.

Clearly $a_{i_0} \notin B_{8\delta + K_x}(q_2)$: since $|y_{i_0}|$ is minimal, i.e. $|y_{i_0}| \leq |a_{i_0} - b_{j'}|$ for every $j = 0, \dots, s$, we get for $j' = 0$ that

$$|y_{i_0}| \leq |a_{i_0} - b_0| = |a_{i_0} - h| \leq d(a_{i_0}, q_2) + |q_2| \leq 8\delta + K_x + |h| < C$$

contrary to (10).

Similarly, $a_{i_0} \notin B_{8\delta + K_x}([h, h x^s])$. Otherwise we may consider a vertex z on $[h, h x^s]$ at distance at most $8\delta + K_x$ from a_{i_0} and choose a vertex z' on p_2 at distance no more than K_x

from z . Finally, there exists a phase vertex $b_{j'}$ on p_2 such that $|b_{j'} - z'| \leq |x|/2$. Using the minimality of $|y_{i_0}|$ we obtain the estimate for the length of phase path:

$$|y_{i_0}| \leq |a_{i_0} - z| + |z - z'| + |z' - b_{j'}| \leq (K_x + 8\delta) + K_x + |x|/2 \leq C,$$

which again contradicts (10). The claim of Step 2 is proved.

Step 3. Let us choose some $|t|, |s| \geq |i_0|$. We choose z on $[e, x^{-t}]$ such that

$$(11) \quad |a_{i_0} - z| \leq K_x.$$

By Step 2 the vertex a_{i_0} is in the set $B_{8\delta+K_x}(q_1)$ and hence $z \in B_{8\delta+2K_x}(q_1)$. Applying lemma 4.1 (ii) to vertices e, x^{-t}, hx^s, z we get using (6), $|h| < 2M_1$:

$$|q_1| \geq |x^t| + |hx^s| - 2|z| - 2\delta - 2(8\delta + 2K_x) \geq |x^t| + (|h| + |x^s| - 4M_1) - 2|z| - 2\delta - 2(8\delta + 2K_x).$$

Inequality (11) implies that $|x^{i_0}| + K_x \geq |z|$ and we conclude that

$$|q_1| \geq |x^t| + |h| + |x^s| - 2(|x^{i_0}| + K_x) + 2(9\delta + 2K_x) = |x^t| + |h| + |x^s| - (4M_1 + 2|x^{i_0}| + 6K_x + 18\delta).$$

It remains to define the constant M_2 (depending only on $\langle x \rangle$, M_1 , \mathcal{H}) to be $4M_1 + 2|x^{i_0}| + 4K_x + 18\delta$ and define $m = |i_0|$. \square

Lemma 4.5. ([Arzh], lemma 13) *Let $n \geq 1$, $r \geq 48\delta$ and elements $h_i, g_i \in G$ ($1 \leq i \leq n$) satisfy :*

$$(12) \quad |g_i| > 15r, \quad (1 \leq i \leq n), \quad |h_1 g_1| \geq |h_1| + |g_1| - 2r,$$

$$(13) \quad |g_{i-1} h_i g_i| \geq |g_{i-1}| + |h_i| + |g_i| - 2r \quad (1 < i \leq n).$$

Then the following assertions are true:

(i) *One has*

$$|h_1 g_1 \dots h_n g_n| \geq |h_1 g_1 \dots h_{n-1} g_{n-1}| + |h_n| + |g_n| - 5r.$$

In particular one has (by induction) $h_1 g_1 \dots h_n g_n \neq e$ in G .

(ii) *Let p be a path in $\text{Cay}(G)$ labeled by $h_1 g_1 \dots h_n g_n$. Then the path p and any geodesic $[p_-, p_+]$ are contained within $4r$ -neighborhood of each other.*

Theorem 4.6. *Let G be a non-elementary hyperbolic group and \mathcal{H} be a K -quasiconvex subgroup of G . Consider an element x in G of infinite order such that $E(x) \cap \mathcal{H} = \{e\}$. Then there exists a number r_0 (depending on \mathcal{H} and x only) such that*

(i) *$(x^s, h) < \frac{r_0}{2}$ for any $h \in H$ and $s \in \mathbb{Z}$ and*

(ii) *for any $r \geq r_0$ there exists $t' > 0$ such that for every $t \geq t'$ and $g = x^t$ the subgroup $\mathcal{H}_1 = \langle g, \mathcal{H} \rangle$ is $(4\delta + 4r + \max\{K, |g|/2\})$ -quasiconvex, of infinite index and canonically isomorphic to $\langle g \rangle * \mathcal{H}$. Moreover, the inequalities of lemma 4.5 hold for r ; $g_i \in \langle g \rangle \setminus \{e\}$ and $h_i \in \mathcal{H}$.*

Proof By lemma 4.3(iii) there exists $M > 0$ such that $(x^s, h) < M$ for any $h \in H$ and $s \in \mathbb{Z}$, hence

$$(14) \quad |hx^s| = |h| + |x^s| - 2(x^s, h) \geq |h| + |x^s| - 2M.$$

Now we consider an arbitrary element $x^{m_1} h x^{m_2}$. If $|h| > 2M + \delta$, then apply lemma 4.1(i) to vertices $e, x^{m_1}, x^{m_1} h, x^{m_1} h x^{m_2}$ in $\text{Cay}(G)$:

$$(15) \quad |x^{m_1} h x^{m_2}| \geq |x^{m_1}| + |h| + |x^{m_2}| - 4M - 2\delta.$$

lemma 2.6(iv) provides a constant $M' \geq 0$ such that the following inequality holds provided $m_1 m_2 \geq 0$:

$$(16) \quad |x^{m_1} x^{m_2}| \geq |x^{m_1}| + |x^{m_2}| - 2M'.$$

By lemma 4.4, there exist a natural number m and a non-negative constant M_2 such that for any $|m_1|, |m_2| \geq m$ the following inequality holds:

$$(17) \quad |x^{m_1} h x^{m_2}| \geq |x^{m_1}| + |h| + |x^{m_2}| - M_2$$

for every $h \neq 1, |h| < 2M + 2\delta$. Now we choose

$$(18) \quad r_0 = \max\{2M + \delta, M_2/2, 48\delta, M'\},$$

and then for any $r \geq r_0$ we choose t' satisfying $|t'| \geq m$ so that the inequality

$$(19) \quad |x^t| > 15r, \text{ holds for every } t \text{ with } |t| \geq |t'|.$$

For $g = x^t$ we consider an arbitrary element $h_1 g^{s_1} \dots h_n g^{s_n}$, where $n \geq 1$, $s_1 \dots s_n \neq 0$ and every $h_i \in \mathcal{H} \setminus \{e\}$ for $i = 1, \dots, n$. We check the first condition of lemma 4.5, i.e. $|g^s| > 15r$. For $s = 1$ it is provided by (19), if $s > 1$ then:

$$|g^s| = |x^{st}| \geq |x^{(s-1)t}| + |x^t| - 2M' > |g^{s-1}| + 13M' > 15r.$$

The second condition of lemma 4.5 is satisfied by (14) and the third because (15)–(17) hold.

We obtain that equality $h_1 g^{s_1} \dots h_n g^{s_n} = e$ in G , where $n \geq 1$, implies that either $s_1 \dots s_n = 0$ or $h_i = e$ for some $i = 1, \dots, n$. Thus the group generated by \mathcal{H} and g is isomorphic to $\mathcal{H} * \langle g \rangle$.

Consider an element $h = h_1 g^{s_1} \dots h_n g^{s_n} h_{n+1}$ in G where $s_1 \dots s_n \neq 0$ and $h_i \neq e$ for $i \leq n$ (h_{n+1} can be the identity e). Define a path pp' in $\text{Cay}(G)$ with p starting from e and label $h_1 g^{s_1} \dots h_n g^{s_n}$ in $\text{Cay}(G)$ and path p' labeled by h_{n+1} . By (H2) and lemma 4.5 (ii) we have that $[e, p'_+] \subset B_{4\delta+4r}(p \cup p')$. In turn, every vertex v of $p \cup p'$ is either within K -neighborhood of $\langle \mathcal{H}, g \rangle$ (if v is a vertex of a subpath labeled by h_i) or at most $|g|/2$ away from $\langle \mathcal{H}, g \rangle$ (if v is a vertex of the subpath labeled by g^{s_i}). We conclude that every vertex $z \in [e, p'_+]$ is within $4\delta + 4r + \max\{K, |g|/2\}$ from a vertex in $\mathcal{H}_1 = \langle \mathcal{H}, g \rangle$. Hence \mathcal{H}_1 is $(4\delta + 4r + \max\{K, |g|/2\})$ -quasiconvex. All conclusions of the theorem are checked for \mathcal{H}_1 except the infiniteness of index. It only remains to observe that the subgroup $\mathcal{H}_2 = \langle \mathcal{H}, g^2 \rangle$ has infinite index in $\langle \mathcal{H}, g \rangle$ and hence in G . It satisfies the all of the conditions of the theorem and hence the conclusion holds for the same constant r and $g = x^{2t}$. \square

Let us consider a path p in $\text{Cay}(G)$ starting at vertex a and ending with b with label $h_1 g^{s_1} \dots h_n g^{s_n} h_n$, i.e. $ah_1 g^{s_1} \dots h_n g^{s_n} h_n = b$ in G where $s_i \in \{\pm 1\}$, $h_i \in \mathcal{H}$ and if $h_i = e$ for $1 < i \leq n$ then $s_{i-1} s_i = 1$. We shall denote:

$$(20) \quad \begin{aligned} a_0 &= a, & b_1 &= ah_1, \\ a_i &= ah_1 \dots h_i g^{s_i} \text{ for } 1 < i \leq n, & b_i &= ah_1 g^{s_1} \dots h_i, \text{ for } 1 < i \leq n. \end{aligned}$$

Lemma 4.7. *Assume that theorem 4.6 holds for \mathcal{H}, x . Take some constant r and an element $g = x^t$ satisfying the same theorem. In the notations (20) we have that $(a, b_{i+1})_{a_i} \leq r + \delta$ for any $i \geq 1$.*

Proof The definition (H3) for a, b_i, a_i, b_{i+1} reads:

$$(21) \quad |a_i - a| + |g^{s_i} h_{i+1}| \leq \max\{|b_i - a| + |h_{i+1}|, |b_{i+1} - a| + |g^{s_i}|\} + 2\delta.$$

The theorem 4.6 permits us to apply the inequalities of lemma 4.5(i) and the second condition in (12) to the left side of (21):

$$|a_i - a| + |g^{s_i} h_{i+1}| \geq (|a_{i-1} - a| + |h_i| + |g^{s_i}| - 5r) + (|g^{s_i}| + |h_{i+1}| - 2r),$$

applying the first inequality in (12) we obtain

$$|a_i - a| + |g^{s_i} h_{i+1}| \geq (|a_{i-1} - a| + |h_i|) + |h_{i+1}| + (2|g^{s_i}| - 7r) > |b_i - a| + |h_{i+1}| + 23r.$$

By the conditions on r in theorem 4.6:

$$(22) \quad |a_i - a| + |g^{s_i} h_{i+1}| > |b_i - a| + |h_{i+1}| + 23r > |b_i - a| + |h_{i+1}| + 2\delta.$$

Hence we may rewrite (22) as $|a_i - a| + |g^{s_i} h_{i+1}| \leq |b_{i+1} - a| + |g^{s_i}| + 2\delta$ and thus (using the second inequality of (12) again):

$$|b_{i+1} - a| \geq |a_i - a| + |g^{s_i} h_{i+1}| - |g^{s_i}| - 2\delta \geq |a_i - a| + |g^{s_i}| + |h_{i+1}| - 2r - |g^{s_i}| - 2\delta = |a_i - a| + |h_{i+1}| - 2\delta - 2r, \text{ which by (H1) implies that } (a, b_{i+1})_{a_i} \leq r + \delta. \square$$

Lemma 4.8. *Assume that theorem 4.6 holds for \mathcal{H} , x . Take some constant r and an element $g = x^t$ satisfying this theorem. In the conventions above (see (20)), assume that*

- (i) $a_i \in B_R$ for some $i > 1$. Then a_1 belongs to $B_{R+2\delta-2r}$.
- (ii) $b_{i+1} \in B_R$ for some $i > 1$. Then a_1 belongs to B_R .

Proof (i) By lemma 4.5(ii) there exists $b' \in [a, a_i]$ such that

$$(23) \quad |b' - a_1| \leq 4r.$$

Using the inequality (12) of the same lemma, we have

$$(24) \quad |b' - a| \geq |a - a_1| - |b' - a_1| \geq |g| + |h_1| - 2r - 4r \geq 9r.$$

Similarly, we may inductively apply lemma 4.5(i) to the subpath of p connecting a_j, a_i for $j < i$:

$$(25) \quad |a_j - a_i| \geq |a_j - a_{i-1}| + |h_i| + |g^{s_i}| - 5r > |a_j - a_{i-1}| + 15r - 5r \geq 10r(i - j),$$

and apply it in order to estimate (for $i > 1$):

$$|b' - a_i| \geq |a_1 - a_i| - |a_1 - b'| \geq 10r(i - 1) - 4r \geq 6r.$$

The inequalities (23) and (24) allow to apply lemma 4.1(iv) to a, a_i, e, b' (with $D = 6r$) and get that $|b' - e| \leq \max\{|a - e|, |a_i - e|\} + 2\delta - 6r$. Since $a, a_i \in B_R$ we have that $|b' - e| \leq R + 2\delta - 6r$. We use the previous inequality together with (23) to conclude that: $|a_1 - e| \leq |a_1 - b'| + |b' - e| \leq 4r + R - 6r + 2\delta = R + 2\delta - 2r$.

(ii) The inequality (25) we have that $|a_i - a| \geq 10r > r + 6\delta + 1$; on the other hand lemma 4.7 implies that $r + 6\delta + 1 \geq (a, b_{i+1})_{a_i} + 5\delta + 1$. Thus we can choose a vertex d on a geodesic $[a, a_i]$ satisfying inequalities:

$$(a, b_{i+1})_{a_i} + 5\delta + 1 \geq |d - a_i| \geq (a, b_{i+1})_{a_i} + 5\delta.$$

Then, by lemma 4.1(iii), d belongs to $B_{4\delta}([a, b_{i+1}])$ and using lemma 4.7 we get

$$d(a_i, [a, b_{i+1}]) \leq |d - b| + 4\delta \leq (a, b_{i+1})_{a_i} + 5\delta + 1 + 4\delta \leq r + 10\delta + 1.$$

By (H2), segment $[a, b_{i+1}]$ belongs to the 4δ -neighborhood of union $[e, a] \cup [e, b_{i+1}]$ which is a subset of B_R because $a, b_{i+1} \in B_R$. Hence

$$|a_i - e| \leq d(a_i, [a, b_{i+1}]) + 4\delta + R \leq R + r + 14\delta + 1$$

and by part (i) of this lemma we conclude that a_1 belongs to $B_{R-r+16\delta+1} \subset B_R. \square$

Lemma 4.9. *Let \mathcal{H} be a K -quasiconvex subgroup in a hyperbolic group G . Assume that $a \in B_R$ and $ah \notin B_R$ for some $h \in \mathcal{H}$. Then either $(a^{-1}, h) \leq 13\delta + K$ or there exists $b_1 \in aH \cap B_R$ such that $b_1 h_1 = ah$ and $|h_1| < |h|$ for some $h_1 \in \mathcal{H}$.*

Proof Assume that $(a^{-1}, h) > 13\delta + K$. We choose a vertex d on the segment $[a, ah]$ such that $|d - a| = K + 8\delta$. By lemma 4.1(iii), $d \in B_{4\delta}([e, a])$ and we can choose $d' \in [e, a]$ to satisfy the inequality $|d - d'| \leq 4\delta$. Then we have

$$\begin{aligned} |d - e| &\leq |e - d'| + |d - d'| \leq (|e - a| - |a - d'|) + 4\delta \leq R - |a - d'| + 4\delta \leq \\ &\leq R - |a - d| + 4\delta + 4\delta \leq R - K. \end{aligned}$$

By quasiconvexity of \mathcal{H} , there exists $b_1 \in a\mathcal{H}$, $|b_1 - d| \leq K$ and hence $b_1 \in B_R$. By the choice of b_1 we have that $b_1^{-1}ah = h_1 \in \mathcal{H}$ and

$$|b_1 - ah| \leq |b_1 - d| + |d - ah| \leq |b_1 - d| + (|a - ah| - |d - a|) \leq K + (|h| - K - 8\delta) < |h|. \square$$

Lemma 4.10. *Assume that theorem 4.6 holds for \mathcal{H} , x . Take some constant r and an element $g = x^t$ satisfying this theorem. We adopt notations (20) and let a, b be vertices in B_R and $ah_1g^sh_2 = b$ in G for some $h_1, h_2 \in \mathcal{H}$, $s \in \{\pm 1\}$. Assume furthermore that $(a^{-1}, h_1) \leq 13\delta + K$ and that $b_1 \notin B_R$. Then*

$$|h_1| \leq K + \frac{r_0}{2} + 15\delta.$$

Proof Definition (H1) and theorem 4.6 yield:

$$(26) \quad \frac{r_0}{2} \geq (a_1, a)_{b_1} \geq \min\{(a, b)_{b_1}, (a_1, b)_{b_1}\} - \delta \geq \min\{(e, a)_{b_1}, (e, b)_{b_1}, (a_1, b)_{b_1}\} - 2\delta.$$

Consider the last two Gromov products on the right-hand side of (26). We have:

$$(e, b)_{b_1} = \frac{1}{2}(|b_1| + |b - b_1| - |b|) = \frac{1}{2}(|b_1| - |b|) + \frac{1}{2}|b - b_1|,$$

by the conditions of this lemma $|b_1| > R \geq |b|$ and using theorem 4.6 we conclude

$$(e, b)_{b_1} \geq 0 + \frac{1}{2}(|g| + |h_2| - r_0) \geq \frac{1}{2}|g| - \frac{r_0}{2} > 7r \geq 7r_0.$$

Similarly,

$$(a_1, b)_{b_1} = \frac{1}{2}(|g| + |b - b_1| - |h_2|) \geq \frac{1}{2}(|g| + (|g| + |h_2| - r_0) - |h_2|) \geq |g| - \frac{r_0}{2} \geq 14\frac{1}{2}r_0.$$

Now we may rewrite (26) as $\frac{r_0}{2} \geq (e, a)_{b_1} - 2\delta$. Note that $(e, b_1)_a = (a^{-1}, h_1) \leq 13\delta + K$, thus by definition of the Gromov product (1):

$$|h_1| = (e, a)_{b_1} + (b_1, e)_a \leq \left(\frac{r_0}{2} + 2\delta\right) + (K + 13\delta). \square$$

In order to estimate the number of \mathcal{H} -cosets in B_R from below, we define $M_R = \{a\mathcal{H} \mid a\mathcal{H} \cap B_R \neq \emptyset\}$ and $Q_R = \{a\mathcal{H} \in M_R \mid \exists b \in B_r, b\mathcal{H} \neq a\mathcal{H} \text{ \& } b\langle \mathcal{H}, g \rangle = a\langle \mathcal{H}, g \rangle\}$.

Lemma 4.11. *Let \mathcal{H} be a free K -quasiconvex subgroup in G . Then for any $k \in \mathbb{N}$ and any $x \in G$ of infinite order either:*

- (i) *there exists $t \neq 0$ such that $x^t \in \mathcal{H}$, or*
- (ii) *there exists t such that for $g = x^t$ the group $\langle \mathcal{H}, g \rangle$ is quasiconvex and canonically isomorphic to the free product $\mathcal{H} * \langle g \rangle$. Moreover $\frac{\#\{Q_R\}}{\#\{B_R\}} \leq \frac{1}{2^k}$ for any $R > 0$.*

Proof Assume that (i) does not hold, so $x^t \in \mathcal{H}$ implies $t = 0$. By lemma 4.2 we have that for every $M \geq 0$ the number of vertices in $B_M(\langle x \rangle) \cap B_M(\mathcal{H})$ is finite. There exists $M_0 \geq 0$ such that $E(x)$ is in M_0 -neighborhood of $\langle x \rangle$, hence $B_M(E(x)) \cap B_M(\mathcal{H}) \subset B_{M+M_0}(\langle x \rangle) \cap B_{M+M_0}(\mathcal{H})$ and hence $\#\{B_M(E(x)) \cap B_M(\mathcal{H})\}$ is finite thus $\#\{E(x) \cap \mathcal{H}\} < \infty$. Since \mathcal{H} is free, the last inequality means that $E(x) \cap \mathcal{H} = \{e\}$.

Using corollary 3.3, we choose c so that that $2^{k+1}(\#\{B_{K+r_0+15\delta}\}^2 \#\{B_{R-c}\}) \leq \#\{B_R\}$ and r according to the theorem 4.6 and satisfying:

$$(27) \quad r \geq \max\{c/7, 2(K + \frac{r_0}{2} + 17\delta)\}.$$

We choose t according to theorem 4.6.

Let $a\mathcal{H}$ belong to Q_R , then there exist $b \in B_R$, $b \notin a\mathcal{H}$, and elements $h_i \in \mathcal{H}$ ($i = 1, \dots, k$) such that $ah_1g^{s_1} \dots g^{s_k}h_{k+1} = b$ in G . By lemma 4.8 we have that either $b_1 = ah_1$ or $a_1 = ah_1g^{s_1}$ or $ah_1g^{s_1}h_2$ belongs to B_R . Hence we can assume that $b = ah_1g^{s_1}h_2$, where

$h_1, h_2 \in \mathcal{H}$, $s \in \{\pm 1\}$. Clearly $a\mathcal{H} \neq b\mathcal{H}$, otherwise $ah_1g^{s_1}h_2 = ah$ and $g^{s_1} = x^{ts_1} \in \mathcal{H}$, contradiction.

We may also assume that a, h_1 are chosen so that $|h_1|$ is minimal with respect to all factorizations $a'h' = ah_1$ in G where $a' \in a\mathcal{H} \cap B_R$. Similarly, we may assume that b, h_2 are chosen so that for any $b' \in b\mathcal{H} \cap B_R$ and $h' \in \mathcal{H}$ the equality $b'h'^{-1} = bh_2^{-1}$ implies that $|h'| \geq |h|$. According to the choice we made, if $h_1 \neq e$ ($h_2 \neq e$) then $b_1 = ah_1 \notin B_R$ (respectively $a_1 = ah_1g^s \notin B_R$). Now we are in position to apply lemma 4.9 to the pairs a, ah_1 and b, bh_2^{-1} , which provides that $(a^{-1}, h_1) \leq K + 13\delta$, $(b^{-1}, h_2^{-1}) \leq K + 13\delta$, and then, by lemma 4.10, we conclude that

$$(28) \quad |h_i| \leq K + \frac{r_0}{2} + 15\delta, \text{ for } i = 1, 2.$$

If $h_i = e$ for $i = 1$ or 2 then the corresponding inequality in (28) holds trivially.

We have that $b_1, a_1 \in B_{R+K+\frac{r_0}{2}+15\delta}$. Since $|g| > 15r$, we can fix a factorization $g = g_1g_2$ in G such that $|g_1| + |g_2| = |g|$ and $|g_1|, |g_2| \geq \frac{15r}{2}$. Let $b' = ah_1g_1$ if $s = 1$ and $b' = ah_1g_2^{-1}$ if $s = -1$, we will call b' a middle point of the path p starting at a with label $h_1g^sh_2$.

Applying lemma 4.1(iv) to vertices b_1, a_1, b' , we obtain that $|b' - e| \leq (R + K + \frac{r_0}{2} + 15\delta) + 2\delta - \frac{15r}{2}$. As we choose r according to (27) we obtain:

$$(29) \quad b' \in B_{R-7r}.$$

We have obtained that if a coset $a'\mathcal{H}$ belongs to Q_R then there exist $a \in a'\mathcal{H} \cap B_R$, $b \in B_R$, $h_1, h_2 \in \mathcal{H}$ and $s \in \{\pm 1\}$ such that the equation $ah_1g^sh_2 = b$ holds in G together with conditions (28) and (29). Hence the number of elements in Q_R is not greater then the number of paths with label $h_1g^sh_2$ in $\text{Cay}(G)$ such that the middle point b' of each path satisfies (29):

$$(30) \quad \# \{Q_R\} \leq \# \{ \text{of } h_1g^sh_2 \text{ satisfying (28)} \} \times \# \{B_{R-7r}\} \leq 2\# \{B_{K+\frac{r_0}{2}+15\delta}\}^2 \times \# \{B_{R-7r}\}.$$

Due to our choice of r in (27) we finally get

$$\# \{Q_R\} \leq 2\# \{B_{K+\frac{r_0}{2}+15\delta}\}^2 \# \{B_{R-7r}\} \leq \frac{1}{2^k} \# \{B_R\}. \square$$

Remark 4.12. Let \mathcal{H} be an infinite quasiconvex subgroup of G of infinite index. Then there exists an element $x \in G$ of infinite order such that it is non-commensurable with any element of \mathcal{H} .

In particular, this remark implies that no infinite index subgroup satisfying the Burnside condition in a non-elementary hyperbolic group is quasiconvex.

Proof By proposition 2.4, for every $N > 0$ there exists a double coset $\mathcal{H}g\mathcal{H}$ which has no representative of length shorter than N . Choose an element g for some $N > 2K + 2\delta$. Then by lemma 2.3 the intersection $\mathcal{H} \cap g^{-1}\mathcal{H}g$ is finite. The subgroup \mathcal{H} contains an element h of infinite order because (by Lemma 18, [IvOl]) every torsion subgroup in a hyperbolic group is finite. The intersection $\langle g^{-1}hg \rangle \cap \mathcal{H} \subset g^{-1}\mathcal{H}g \cap \mathcal{H}$ is finite and hence $x = g^{-1}hg$ is non-commensurable with any element of \mathcal{H} . \square

Theorem 4.13. For every non-elementary δ -hyperbolic group G and any $0 < q < 1$ there exists a free subgroup H satisfying the Burnside condition and such that $\frac{\# \{a\mathcal{H} \mid a\mathcal{H} \cap B_R \neq \emptyset\}}{\# \{B_R\}} \geq q$.

Proof We choose a sequence $\{k_i\}_{i \in \mathbb{N}}$ such that

$$(31) \quad \sum_{i=1}^{\infty} \frac{1}{2^{k_i}} < 1 - q.$$

Let $\{x_j\}$, $j \in \mathbb{N}$ be a list of all elements of infinite order in G . We fix notations $\mathcal{H}_i = \langle x_1^{t_1}, \dots, x_i^{t_i} \rangle$ for some positive numbers $t_i \in \mathbb{N}$ which we will determine later. We define $H = \cup_{i=1}^{\infty} \mathcal{H}_i$, it clearly satisfies the Burnside condition. Then we denote $M_R^i = \{a\mathcal{H}_i \mid a\mathcal{H}_i \cap B_R \neq \emptyset\}$ and $Q_R^i = \{a\mathcal{H}_i \mid \exists b \in B_R \text{ such that } a\mathcal{H}_i \neq b\mathcal{H}_i \text{ \& } a\mathcal{H}_{i+1} = b\mathcal{H}_{i+1}\}$.

We set $\mathcal{H}_0 = \{e\}$ and thus $M_R^0 = B_R$. lemma 4.11 (applied to \mathcal{H}_0 , x_1 and k_1) provides that there exists $t_1 > 0$ such that $\mathcal{H}_1 = \langle g_1 \rangle$ (where $g_1 = x_1^{t_1}$) is cyclic, quasiconvex and $\frac{\# \{Q_R^0\}}{B_R} \leq \frac{1}{2^{k_1}}$ for any $R > 0$. It provides the following estimate for M_R^1 :

$$\# \{M_R^1\} \geq \# \{M_R^0\} - \# \{Q_R^0\} \geq (1 - \frac{1}{2^{k_1}}) \# \{B_R\}.$$

Now we assume by induction that a free quasiconvex subgroup $\mathcal{H}_i = \langle x_1^{t_1}, \dots, x_i^{t_i} \rangle$ has been constructed by repeated application of lemma 4.11 and M_R^i satisfies inequality

$$(32) \quad \# \{M_R^i\} \geq (1 - \frac{1}{2^{k_1}} - \frac{1}{2^{k_2}} - \dots - \frac{1}{2^{k_i}}) \# \{B_R\}.$$

If $\langle x_{i+1} \rangle \subset B_M(\mathcal{H}_i)$ for some $M \geq 0$ then by lemma 4.3 there exists $t_{i+1} > 0$ such that $x_{i+1}^{t_{i+1}} \in \mathcal{H}_i$ and we can set $\mathcal{H}_{i+1} = \mathcal{H}_i$, finishing the induction step ($M_R^{i+1} = M_R^i$).

Assume now that $\langle x_{i+1} \rangle \not\subset B_M(\mathcal{H}_i)$ for any non-negative M , then by lemma 4.3 we have $\# \{E(x) \cap \mathcal{H}_i\} < \infty$ and hence (because \mathcal{H}_i is free) $E(x) \cap \mathcal{H}_i = \{e\}$. We choose t_{i+1} applying lemma 4.11 to $\mathcal{H}_i, x_{i+1}, k_{i+1}$ and using the induction assumption (32):

$$\begin{aligned} \# \{M_R^{i+1}\} &\geq \# \{M_R^i\} - \# \{Q_R^i\} \geq (1 - \frac{1}{2^{k_1}} - \frac{1}{2^{k_2}} - \dots - \frac{1}{2^{k_i}}) \# \{B_R\} - \# \{Q_R^i\} \geq \\ &\geq (1 - \frac{1}{2^{k_1}} - \frac{1}{2^{k_2}} - \dots - \frac{1}{2^{k_{i+1}}}) \# \{B_R\}, \end{aligned}$$

and by (31):

$$\# \{M_R\} \geq (1 - \sum_{i=1}^{\infty} \frac{1}{2^{k_i}}) \# \{B_R\} > q \# \{B_R\}. \square$$

Proof of theorem 3.8 By the remark 3.9, the number of left cosets intersecting B_n is equal to the number $f_{H \setminus G}(n)$ of right ones. We can now fix some $0 < q < 1$ and using theorem 4.13 find a group H such that $f_{H \setminus G}(n) \geq qf(n)$. Thus (by remark 3.4) the growth of action of G on $H \setminus G$ is maximal. \square

5. PROOF OF THEOREM 1.2 AND COROLLARY 1.3

Lemma 5.1. *Let G be a non-elementary hyperbolic group and \mathcal{H} be a quasiconvex subgroup in G such that $E(G) \cap \mathcal{H} = \{e\}$, then there exists $g \in G$ of infinite order such that $E(g) = \langle g \rangle \times E(G)$ and $\langle \mathcal{H}, g \rangle$ is quasiconvex of infinite index and is canonically isomorphic to $\mathcal{H} * \langle g \rangle$.*

Proof By lemma 2.7 there exists $y \in G$ of infinite order such that $E(y) = \langle y \rangle \times E(G)$. We have either

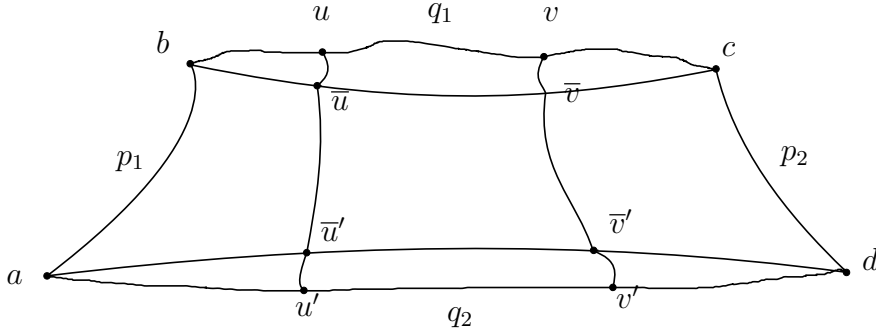
- (I) $\# \{E(y) \cap \mathcal{H}\} < \infty$,
- or (II) $\# \{E(y) \cap \mathcal{H}\}$ is infinite.

Take an element $y^k a \in E(g)$ and assume $y^k a \in \mathcal{H}$ for some $k \in \mathbb{Z}$ and $a \in E(G)$, then we have $(y^k a)^n = y^{kn} a^n$ for every n (because a commutes with y). Note that for a non-zero k the equality $y^{kn_1} a^{n_1} = y^{kn_2} a^{n_2}$ holds in G if and only if $n_1 = n_2$.

Hence in case (I) we have that $k = 0$ and thus $E(y) \cap \mathcal{H} \subset E(G) \cap \mathcal{H} = \{e\}$ and we may apply theorem 4.6 to find $t > 0$ and obtain the canonical isomorphism $\langle \mathcal{H}, y^t \rangle \cong \mathcal{H} * \langle y^t \rangle$.

Thus we only need to consider case (II) when $y^k a \in \mathcal{H}$ for some non-zero k . Replacing y with $y^k a$ we may assume that y is in \mathcal{H} . By remark 4.12 there exists an element x of infinite order such that x is non-commensurable with any element of \mathcal{H} . Replacing x with

FIGURE 2.



its non-zero power if necessary, we may assume that x commutes with $E(G)$. We define a subgroup $\mathcal{H}_1 = \langle y \rangle$, which is quasiconvex by lemma 2.6(ii). Since $\mathcal{H}_1 \cap E(x) = \{e\}$, there exists a constant $r_0 \geq 0$ such that $\langle x^t, y^s \rangle < \frac{r_0}{2}$ for all $t, s \in \mathbb{Z}$ by theorem 4.6. By part (ii) of the same theorem and $r = r_0$, there exists $t' > 0$ such that $\langle x^{t'}, y \rangle \cong \langle x^{t'} \rangle * \langle y \rangle$. We denote $x_1 = x^{t'}$ so the subgroup $\langle x_1, y \rangle$ is free quasiconvex and inequalities of lemma 4.5 hold for \mathcal{H}_1, x_1 and $r = r_0$. In particular, for every reduced word w in $\langle x_1, y \rangle$, the corresponding path in $\text{Cay}(G)$ with label w is within $4r_0$ -neighborhood of a geodesic connecting its ends.

By lemma 4.3 there exists $M \geq 0$ such that $\langle x^s, h \rangle < M$ for every $s \in \mathbb{Z}$, $h \in \mathcal{H}$. Choose $t > 0$ such that $|x_1^s| > 4r_0 + K + 2M$ for every $|s| \geq t$ and denote $x_2 = x_1^t$. We have that for any non-zero m :

$$(33) \quad d(x_2^m, h) \geq |x_2^m| + |h| - 2M \geq |x_2^m| - 2M > 4r_0 + K.$$

Let element $w = y^{s_0} x_2^{t_1} y^{s_1} \dots x_2^{t_n} y^{s_n}$ satisfy $s_i, t_i, t_n \neq 0$ for $i = 1, \dots, n-1$ and assume that $w \in \mathcal{H}$. Then every phase vertex of w is in $(4r_0 + K)$ -neighborhood of \mathcal{H} , which contradicts inequality (33) because $d(y^{s_0} x_2^{t_1}, y^{s_0}) = d(x_2^{t_1}, e) > 4r_0 + K$. We conclude that an element w of the free group $\mathcal{H}_2 = \langle x_2, y \rangle$ is commensurable with an element of \mathcal{H} if and only if $w = y^s$ for some integer s .

Now we consider an element $y^k x_2^k$ for sufficiently large k and will show that the group $E(y^k x_2^k)$ is equal to $\langle y^k x_2^k \rangle \times E(G)$. Let z be an element of $E(y^k x_2^k)$, i.e. the equality $z(y^k x_2^k)^m z^{-1} = (y^k x_2^k)^{m'}$ holds in G for some $m = \pm m' \neq 0$. We choose a constant

$$M_0 > 2|z| + 8r_0 + 26\delta + (3k+1)(\max\{|y|, |x_2|\})$$

and a natural number s divisible by m such that $|(y^k x_2^k)^s| \geq M_0$. We consider a closed path $p_1 q_1 p_2 q_2$ in $\text{Cay}(G)$ such that $\text{lab}(p_1) = \text{lab}(p_2) = z$, $\text{lab}(q_1) = (y^k x_2^k)^s$, and $\text{lab}(q_2^{-1}) = (y^k x_2^k)^{s'}$, where $s' = \pm s$. For convenience we denote the initial vertices of p_1, q_1, p_2, q_2 by a, b, c, d respectively. We choose a vertex \bar{u} on $[b, c]$ at distance $|z| + 5\delta$ in $\text{Cay}(G)$ from the vertex b . Then, using (H2), \bar{u} is in 4δ -neighborhood of some u_1 on $[a, c] \cup [a, b]$ and, by the choice of \bar{u} , is actually on $[a, c]$. Using (H2) again, and taking into account that

$$|\bar{u}_1 - c| \geq |b - c| - |z| - 5\delta - 4\delta \geq M_0 - |z| - 9\delta > |z| + 5\delta$$

we obtain that there exists \bar{u}' on $[a, d]$ satisfying $|\bar{u}' - u_1| \leq 4\delta$. Hence

$$(34) \quad |\bar{u}' - \bar{u}| \leq 8\delta.$$

Similarly we can choose \bar{v} on $[b, c]$ at distance $|z| + 5\delta$ from the vertex c and a vertex \bar{v}' on $[a, d]$ such that

$$(35) \quad |\bar{v}' - \bar{v}| \leq 8\delta.$$

Since q_1 and $[b, c]$ are within $4r_0$ -neighborhood of each other, we find phase vertices u, v on q_1 relative to the factorization $y^k x_2^k \dots y^k x_2^k$ of $lab(q_1)$ such that

$$(36) \quad |u - \bar{u}|, |v - \bar{v}| \leq 4r_0 + \frac{1}{2} \max\{|y|, |x_2|\}.$$

Similarly we find phase vertices u', v' on q_2 such that

$$(37) \quad |u' - \bar{u}'|, |v' - \bar{v}'| \leq 4r_0 + \frac{1}{2} \max\{|y|, |x_2|\}.$$

Now we consider a closed path $p'_1 q'_1 p'_2 q'_2$, where q'_1, q'_2 are subpaths of q_1 and q_2 respectively and $p'_1 = [u', u]$, $p'_2 = [v', v]$. According to inequalities (34)–(37) above:

$$(38) \quad |p_i| \leq 8r_0 + \max\{|y|, |x_2|\} + 8\delta, \text{ where } i = 1, 2.$$

Note that

$$|q'_1| = |u - v| \geq |\bar{u} - \bar{v}| - |u - \bar{u}| - |v - \bar{v}| \geq |q_1| - |\bar{u} - c| - |\bar{v} - c| - |u - \bar{u}| - |v - \bar{v}|,$$

and using the definitions of \bar{u}, \bar{v} and (36) we get:

$$(39) \quad |q'_1| \geq M_0 - 2|z| - 10\delta - 8r_0 - \max\{|y|, |x_2|\} > 3k \max\{|y|, |x_2|\}.$$

We consider $q'_1 = t_1 \dots t_n$, where either $lab(t_{2i-1}) = y^k$, $lab(t_{2i}) = x_2^k$ for every $1 < 2i \leq n-1$ or $lab(t_{2i-1}) = x_2^k$, $lab(t_{2i}) = y^k$ for every $1 < 2i \leq n-1$. By the estimate on q'_1 above we have that $n \geq 4$.

Now we use (34), (35) and (39) to obtain:

$$|q'_2| = |u' - v'| \geq |\bar{u}' - \bar{v}'| - |u' - \bar{u}'| - |v' - \bar{v}'| \geq |\bar{u} - \bar{v}| - |\bar{u}' - \bar{u}| - |\bar{v}' - \bar{v}| - |u' - \bar{u}| - |v' - \bar{v}| \\ \geq M_0 - 16\delta - 2|z| - 10\delta - 8r_0 - \max\{|y|, |x_2|\} > 3k \max\{|y|, |x_2|\}.$$

We consider $(q'_2)^{-1} = t'_1 \dots t'_{n'}$, where $lab(t_{2i}) = y^{k'}$, $lab(t_{2i+1}) = x_2^{k'}$ for every $1 < 2i < n' - 1$ or $lab(t_{2i}) = x_2^{k'}$, $lab(t_{2i+1}) = y^{k'}$ for every $1 < 2i < n' - 1$ where $k = \pm k'$. By the estimate on q'_1 above, $n' \geq 4$.

We can now apply lemma 2.10 to the closed path $p'_1 q'_1 p'_2 q'_2$ with upper bound on $|p'_i|$ provided by (38) and obtain a constant m_0 such that for every $k \geq m_0$ the paths t_2 and t_3 are compatible with t'_i and t'_{i+1} respectively (for some unique i). Let us denote for convenience $lab(t_2) = W_2^k$, $lab(t_3) = W_3^k$, $lab(t'_i) = \bar{W}_i^{k'}$, $lab(t'_3) = \bar{W}_{i+1}^{k'}$, where the sets $\{W_2, W_3\}$, $\{\bar{W}_i, \bar{W}_{i+1}\}$ and $\{x, y\}$ are all equal. Lemma 2.10 also provides that there exist compatibility paths v_2 and v_3 with labels V_2, V_3 such that:

$$(40) \quad V_2^{-1} W_2^r V_2 = \bar{W}_i^s, \quad V_3^{-1} W_3^{r'} V_3 = \bar{W}_{i+1}^{s'}$$

for some $r, s, r', s' > 0$. Because x_2 and y are non-commensurable, the equalities (40) are only possible if $W_2 \equiv \bar{W}_i^{\pm 1}$ and $W_3 \equiv \bar{W}_{i+1}^{\pm 1}$. Moreover, one of the exponents is positive because y is not conjugate with y^{-1} and thus $lab(q_2)^{-1} = (y^k x_2^k)^m$ and $W_2 \equiv \bar{W}_i$ and $W_3 \equiv \bar{W}_{i+1}$. Now by definition of compatible paths we have that $V_2 \in E(W_2)$ and $V_3 \in E(W_3)$. Consider a path v connecting the terminal vertex of t_2 with the terminal vertex of t'_i . We also consider a pair of paths $\bar{q}_1 v_2 \bar{q}_2$ and $\bar{q}_3 v_3 \bar{q}_4$ each of which has the same initial and terminal vertices as the path v . Reading off their labels provides the following inequalities in G :

$$lab(v) = W_2^{s_1} V_2 W_2^{s_2} = W_3^{s_3} V_3 W_3^{s_4}$$

for some exponents $s_i \in \mathbb{Z}$. Hence $lab(v) \in E(W_2) \cap E(W_3) = E(x_2) \cap E(y) = E(G)$. We obtained that $z = lab(p_1)$ is equal to either $(y^k x_2^k)^{s'} lab(v)^{-1} (y^k x_2^k)^{-s''}$ or $(y^k x_2^k)^{s'} y^k lab(v)^{-1} ((y^k x_2^k)^{s''} y^k)^{-1} = (y^k x_2^k)^{s'} lab(v)^{-1} (y^k x_2^k)^{-s''}$ for some non-negative numbers s', s'' . In both cases $z \in E(G) \times \langle (y^k x_2^k) \rangle$.

We obtained that $E(g) \cap \mathcal{H} = \{e\}$ for $g = y^k x_2^k$, and now the lemma follows from theorem 4.6. \square

Proof of theorem 1.2 (1) The sufficiency is provided by theorem 4.6. Assume that there exist an element x of infinite order and $t \neq 0$ such that $\langle \mathcal{H}, x^t \rangle \cong \mathcal{H} * \langle x^t \rangle$. Take $h \in E(x) \cap \mathcal{H}$, then there exist $n \neq 0$ and $n' = \pm n$ such that $h^{-1} x^n h x^{n'} = e$ in G . Thus $h^{-1} x^{tn} h x^{tn'} = e$ in G which implies $h = e$.

(2) The sufficiency follows from lemma 5.1. To show the necessity it is enough to notice that if the element x satisfying part (1) exists then $E(G) \cap \mathcal{H} \leq E(x) \cap \mathcal{H} = \{e\}$.

(3) Denote the subgroup $E(G) \cap (\mathcal{H} * \langle x^t \rangle)$ by K , it is a finite subgroup in $\mathcal{H} * \langle x^t \rangle$. By Kurosh subgroup theorem, K is conjugate to a subgroup in \mathcal{H} . On the other hand K is normal in $\mathcal{H} * \langle x^t \rangle$ and thus $K < \mathcal{H}$, i.e. $E(G) \cap \mathcal{H} * \langle x^t \rangle \leq E(G) \cap \mathcal{H} = \{e\}$

Proof of corollary 1.3

Consider a canonical homomorphism $\phi : G \rightarrow \bar{G} = G/E(G)$. It is clear that $E(\bar{G}) = \{e\}$: the subgroup $E(\bar{G})$ is finite normal, hence the subgroup $\phi^{-1}(E(\bar{G}))$ is finite normal and thus $\phi^{-1}(E(\bar{G})) \leq E(G)$. Homomorphism ϕ is a quasi-isometry because $E(G)$ is finite. Thus $\bar{\mathcal{H}} = \phi(\mathcal{H})$ is quasiconvex of infinite index in \bar{G} . We can apply lemma 5.1 to $\bar{\mathcal{H}}, \bar{G}$ and find some \bar{y} such that $E(\bar{y})$ is infinite cyclic and obtain the isomorphism $\langle \mathcal{H}, y \rangle \cong \bar{\mathcal{H}} * \langle \bar{y} \rangle$. Consider some preimage y of \bar{y} . $\phi^{-1}(\langle \bar{\mathcal{H}}, \bar{y} \rangle) = \langle \mathcal{H} \cdot E(G), y \rangle \cong \mathcal{H} \cdot E(G) *_{E(G)} \langle y, E(G) \rangle$. \square

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REFERENCES

- [Arzh] *G. N. Arzhantseva*, On Quasiconvex Subgroups of Word Hyperbolic Groups. *Geometriae Dedicata* 87: 191–208, 2001.
- [AL] *G. N. Arzhantseva, I. G. Lysenok* Growth tightness for word hyperbolic groups. *Math. Z.* 241 (2002), no. 3, 597611.
- [BO] *Y. Bahturin, A. Olshanskiy* Actions of maximal growth. *Proc. Lond. Math. Soc.* (3) 101 (2010), no. 1, 2772.
- [CDP] *M. Coornaert, T. Delzant et A. Papadopoulos*, *Geometrie et Theorie des Groupes, les Groupes Hyperboliques de Gromov*, Lecture Notes in Math., n 1441, Springer, 1990.
- [Ghys] *E. Ghys, P. de la Harpe*, *Sur Les Groupes Hyperboliques D'apres Mikhael Gromov*. Birkhäuser, 1990.
- [GMRS] *R. Gitik, M. Mitra, E. Rips, M. Sageev*, Widths of subgroups. *Trans. Amer. Math. Soc.* 350 (1998), no. 1, 321329.
- [Gre] *F.P. Greenleaf*, *Invariant Means on Topological Groups and Their Applications*, Van Nostrand Reinhold (1969).
- [Gro] *M. Gromov*, *Hyperbolic Groups. Essays in group theory*, MSRI Publications, Springer, 1987.
- [IvOl] *S. V. Ivanov, A. Yu. Olshanskiy*, Hyperbolic groups and their quotients of bounded exponents. *Trans. Amer. Math. Soc.* 348 (1996), pp 2091–2138.
- [Mack] *T. P. Mack*, *Quasiconvex Subgroups and Nets in Hyperbolic Groups*. PhD Thesis, CalTech 2006. <http://thesis.library.caltech.edu/2461/1/thesis.pdf>
- [M-P] *Eduardo Martinez-Pedroza* On residual properties of word hyperbolic groups, *GEOMETRIAE DEDICATA*, to appear, <http://www.springerlink.com/content/y70t6026j8486241>.
- [Min] *A. Minasyan* On residual properties of word hyperbolic groups, *J. of Group Theory* 9 (2006), No. 5, pp. 695–714.
- [LSch] *M. Lyndon, P. Schupp*, *Combinatorial group theory*. Springer, 1977.
- [Olsh93] *A. Yu. Olshanskiy*, On residualizing homomorphisms and G-subgroups of hyperbolic groups. *IJAC*, Vol.3, No.4, 1993.

[Sta] *J. R. Stallings*, Topology of finite graphs. *Invent. Math.*, 71 (1983), 551–565.